

Mathematical modeling of economic processes

Modelización matemática de los procesos económicos

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ABSTRACT:

We study the optimization of production output as a process of finding the conditional maximum of the Cobb-Douglas production function in the presence of the budget constraint. This function is often used in the analysis of economic development. The use of the Lagrange method allows obtaining a stationary point of the Lagrange function. The examination of this point reveals the presence of a conditional maximum at it. The obtained coordinates of the point of conditional maximum of the output volume mean that, under the optimal functioning of the firm, i.e., such functioning, for which the greatest production output is achieved with a fixed budget, the budget should be distributed between fixed assets and labor resources in proportion to the corresponding elasticities of the production output. These coordinates allow finding the values of the largest production output and the marginal maximum output with respect to the budget.

Keywords: production function, conditional maximum, budget constraint, Lagrange multiplier, output elasticity.

RESUMEN:

Estudiamos la optimización de la producción como un proceso para encontrar el máximo condicional de la función de producción de Cobb-Douglas en presencia de la restricción presupuestaria. Esta función se utiliza a menudo en el análisis del desarrollo económico. El uso del método de Lagrange permite obtener un punto estacionario de la función de Lagrange. El examen de este punto revela la presencia de un máximo condicional en él. Las coordenadas obtenidas del punto de máxima condicional del volumen de salida significan que, bajo el óptimo funcionamiento de la empresa, es decir, tal funcionamiento, para lo cual la mayor producción productiva se consigue con un presupuesto fijo, el presupuesto debe ser distribuido entre los activos fijos y los recursos laborales en proporción a las elasticidades correspondientes de la producción. Estas coordenadas permiten encontrar los valores de la producción más grande y la salida máxima marginal con respecto al presupuesto.

Palabras clave: función de producción, máximo condicional, restricción presupuestaria, multiplicador de Lagrange, elasticidad de salida.

1. Introduction

The transition to market economy requires that enterprises increase the production efficiency and the competitiveness of products and services through introduction of the results of scientific and technological progress, effective forms of economic and production management, overcoming mismanagement, promotion of entrepreneurship, initiative, etc.

An important role in solving this problem belongs to the economic analysis of the activity of economic entities. With its help, the strategy and tactics of enterprise development are worked out, plans and management decisions are justified, control over their implementation is realized, the reserves for increasing production efficiency are identified, and the performance results of the enterprise, its departments and employees are evaluated (Fomin 2009).

The technical and organizational level of production determines the volumes of production and the sales of products, which, in turn, serve as the basis for the formation of the main type of income, the sales proceeds. The growth in the volume of production and the increase in its quality directly affect the value of production costs, profit and profitability of the enterprise. The activities of economic entities should be aimed at producing the maximum quantity of high-quality products at minimal cost. Therefore, the analysis of the work of enterprises begins with the study of output indicators. The analysis of the production and sale of products allows assessing the competitive position of the enterprise and its ability to respond flexibly to changing market conditions.

Among social sciences, economics uses mathematics to the greatest extent. To work successfully, a modern economist must not only have good basic knowledge, but also be able to use mathematical methods to solve practical problems (Cherkunova 2011). The implementation of radical reforms in the transition to a market economy requires carrying out mathematical calculations of the future results of economic activity. The modern economy uses almost the entire apparatus of applied mathematics (Krass, and Chuprynov 2008; Krass, 2003).

Mathematical calculations are an effective means of improving management of economic activity (Cherkunova, and Novoselova, 2011); without them it is impossible to achieve good economic results. The discovered regularities related to economy are not only of theoretical value; they are widely used in practice: in planning, management and forecasting (Kremer 2002).

The present work is devoted to studying the production output optimization. This topic is *of great current interest*, since the growth rates of the volume of production and sales of products directly affect the value of costs, profit and profitability of the company, serve as the basis for the formation of the main type of its income, the sales revenue, and allow assessing its competitive position.

The *purpose* of this work is to investigate the output function in terms of the existence of a conditional maximum with the help of a system of corresponding differential equations and to find, with the help of this system, the maximum volume of the company's output under a budgetary constraint, and the values of the largest and the marginal maximum production output with respect to the budget.

2. Application of the Lagrange method to optimize production output

The majority of the most important concepts of economics are examples of standard concepts of mathematics. Let us consider the optimization of production output as a process of finding its maximum, i.e., choosing the best option from the set of possible ones (Braun, 1983).

As an example, we choose the Cobb-Douglas model (Kremer and Putko 2003), which is often used in the analysis of economic development. The production function in this model is a function that shows the production output Y for the capital input K and labor input L . For the

case of two variables, it has the form

$$Y = AK^\alpha L^{1-\alpha} \quad (1),$$

where $A > 0$ is the parameter of productivity of a particular technology, $0 < \alpha < 1$ is the proportion of capital in the income. If K and L are taken in money terms, then the budget constraint has the form

$$K + L = M \quad (2),$$

where M is the company's budget. For the Cobb-Douglas function, the elasticity of production output with respect to capital is equal to α and with respect to labor is equal to $1 - \alpha$ (Dougherty 1992).

3. Finding the parameters of the model

Let us find the conditional maximum of the production function (1) under the budget constraint (2). The function $Y = Y(K, L)$ reaches a maximum only at stationary points. We find these points by the Lagrange method; to this end, we introduce a new function, which is called the Lagrange function

$$F(K, L, \lambda) = Y(K, L) - \lambda(K + L - M),$$

where λ is an indefinite Lagrange multiplier, which is regarded as a new variable (Kremer, and Putko 2003).

Let us find the stationary points of the Lagrange function as a solution of the system of equations

$$\begin{cases} \frac{\partial F}{\partial L} = A(1-\alpha)K^\alpha L^{-\alpha} - \lambda = 0 \\ \frac{\partial F}{\partial L} = -(K + L - M) = 0 \\ \frac{\partial F}{\partial K} = A\alpha K^{\alpha-1} L^{1-\alpha} - \lambda = 0 \end{cases} .$$

We obtain a stationary point (K_0, L_0, λ_0) of the Lagrange function $F(K, L, \lambda)$, where

$$K_0 = \alpha M, L_0 = (1 - \alpha)M, \lambda_0 = A\alpha^\alpha (1 - \alpha)^{1-\alpha} \quad (3).$$

The first two numbers (K_0, L_0) give the coordinates of the conditional maximum of the output Y , if this maximum is present.

Let us verify the existence of a conditional maximum in our case; to this end, we denote

$$P = \frac{\partial^2 F}{\partial K^2}(K_0, L_0, \lambda_0), Q = \frac{\partial^2 F}{\partial L^2}(K_0, L_0, \lambda_0), R = \frac{\partial^2 F}{\partial \lambda^2}(K_0, L_0, \lambda_0).$$

When $PQ - R^2 > 0$, the point (K_0, L_0) is a point of conditional extremum of the function Y with the budget constraint equation (2). In this case, if $P < 0$, then (K_0, L_0) is the point of conditional maximum.

In our case, we have

$$P = -A(1-\alpha)^{2-\alpha} \alpha^{\alpha-1} M^{-1},$$

$$Q = -A\alpha^{\alpha+1} (1-\alpha)^{-\alpha} M^{-1}, R = 0,$$

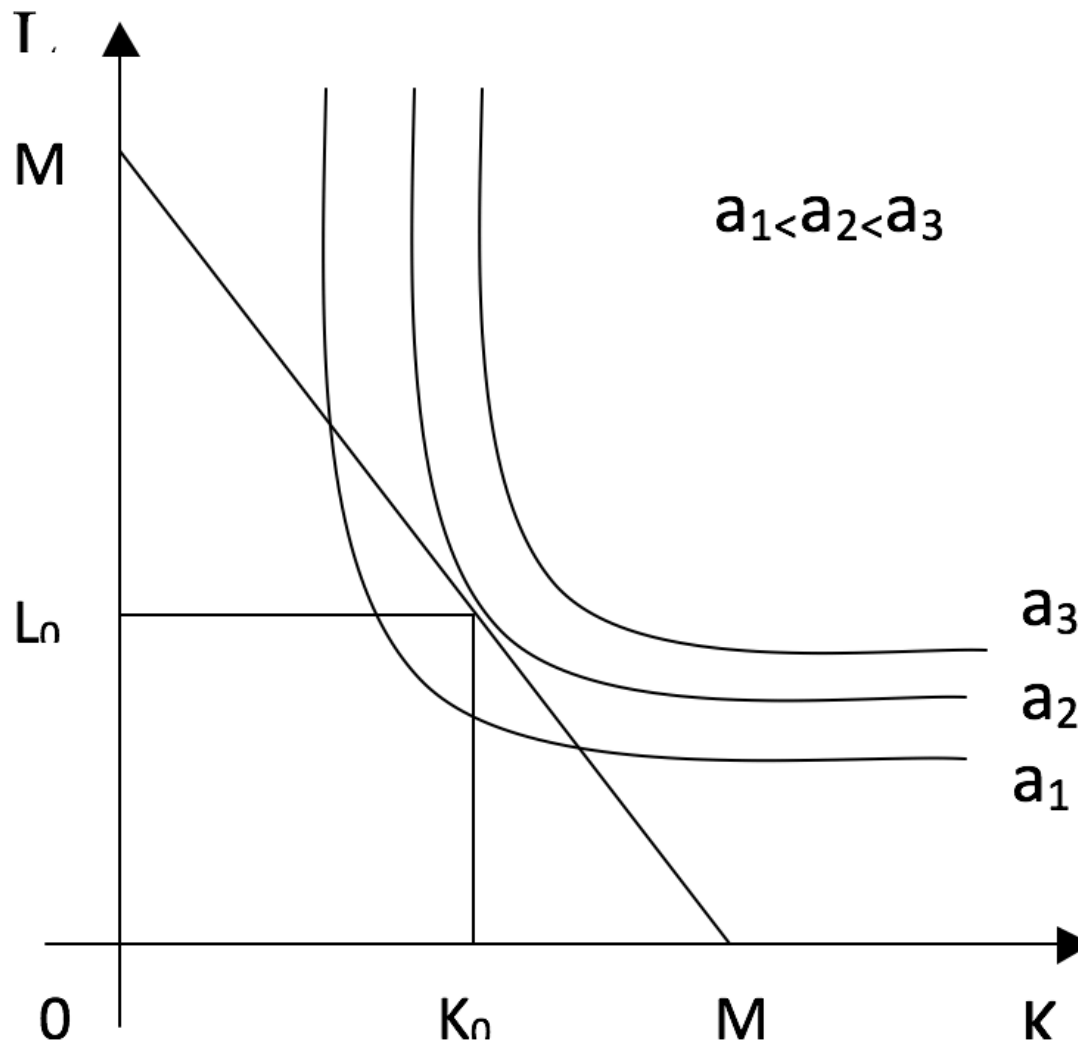
$$PQ - R^2 = A^2 \alpha^{2\alpha} (1-\alpha)^{2-2\alpha} M^{-2} > 0$$

hence, (K_0, L_0) is a point of conditional extremum.

Since $P = -A(1-\alpha)^{2-\alpha} \alpha^{\alpha-1} M^{-1} < 0$, (K_0, L_0) is a point of conditional maximum of the function of the output volume $Y = AK^\alpha L^{1-\alpha}$ with the budget constraint equation $K + L = M$.

Fig. 1 shows the isoquants ($a_1 = Y(K, L), a_2 = Y(K, L), a_3 = Y(K, L)$) of the function $Y = Y(K, L)$ and the straight line of the budget constraint $K + L = M$.

Figure 1. Isoquants of the Cobb-Douglas production function.



The isoquant (a_2 in Figure 1), which just touches the straight line, is located farthest from the origin among the isoquants having common points with the straight line; thus, it corresponds to the largest production output. The quantities K_0 and L_0 are optimal from the point of view of maximum production output under the cost limitation equal to M .

4. Results

The obtained equation (3) means that, under the optimal functioning of the firm (i.e., such

functioning that achieves the highest production output under a fixed budget), one should distribute the budget M between the fixed assets K and the labor resources L in proportion to the elasticities of output with respect to K and L . If we denote by Y_0 the value of the largest production output

$$Y_0 = Y(K_0; L_0) = Y(\alpha M, (1-\alpha)M) = A\alpha^\alpha (1-\alpha)^{1-\alpha} M = \lambda_0 M,$$

then the marginal maximum output with respect to budget will be

$$\frac{dY_0}{dM} = \frac{d}{dM}(\lambda_0 M)' = \lambda_0,$$

Consequently, the Lagrange multiplier is equal to the marginal maximum production output with respect to budget, i.e., if the budget is increased by 1, then the largest output will increase by λ_0 units.

5. Discussion of the results and their application

1. The use of the Lagrange method allows obtaining a stationary point (K_0, L_0, λ_0) of the Lagrange function $F(K, L, \lambda) = Y(K, L) - \lambda(K + L - M)$, where

$$K_0 = \alpha M, \quad L_0 = (1-\alpha)M, \quad \lambda_0 = A\alpha^\alpha (1-\alpha)^{1-\alpha}.$$

The first two numbers (K_0, L_0) provide the coordinates of the conditional maximum of the output volume Y , if this maximum is present.

2. The performed examination showed the existence of a conditional maximum of the function Y at this point (K_0, L_0) .
3. The obtained coordinates

$$K_0 = \alpha M, \quad L_0 = (1-\alpha)M$$

of the point of conditional maximum mean that, under the optimal functioning of the firm (i.e., such a functioning, for which the greatest production output is achieved with a fixed budget), the budget M should be distributed between fixed assets K and labor resources L in proportion to the elasticities of the production output with respect to K and L .

4. The values of the largest production output and the marginal maximum output with respect to budget are equal to the values of the functions Y and $\frac{dY}{dM}$ at this point

$$(K_0, L_0):$$

$Y_0 = Y(K_0; L_0) = \lambda_0 M$ is the value of the largest output with respect to the budget,

$\frac{dY_0}{dM} = \lambda_0$ is the value of the marginal maximum output with respect to the budget.

Hence, the Lagrange multiplier $\lambda_0 = A\alpha^\alpha (1-\alpha)^{1-\alpha}$ equals the marginal maximum production output with respect to budget, i.e., if the budget is increased by 1, then the maximum output will increase by λ_0 units.

6. Conclusion

In the work, the function of production output was investigated, which directly affects the value of costs, profit, profitability and competitive position of the firm, whose activities should be aimed at producing the maximum quantity of high-quality products at minimal costs.

As this function, the production function Y of the Cobb-Douglas model was considered, which is often used in the analysis of the economic development. The function Y of this model expresses the dependence of the volume of output on the input of capital K and labor resources L .

To find the conditional maximum of the function Y , a stationary point (K_0, L_0, λ_0) was found by the Lagrange method as a solution of the formed system of equations, where λ is the Lagrange multiplier. The examination of this point showed the presence at it of a conditional maximum of the output volume Y under the budget constraint $K + L = M$, where M is the budget of the firm.

The obtained coordinates (K_0, L_0) of the point of the conditional maximum mean that, under the optimal functioning of the firm (i.e., such functioning, which achieves the largest output under a fixed budget), the budget M should be distributed between the fixed assets K and the labor resources L in proportion to the elasticities of output with respect to K and L .

These coordinates allowed finding the values of the largest production output and the marginal maximum output with respect to the budget.

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